Notes on Stratified spaces.

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1 Stratified space

1.1 Some topological notions

Definition 1. Let X be a topological space and A be its subspace with subspace topology. A is called

- *locally closed* if A is open in its closure;
- *locally compact* if each point $a \in A$ has a *relatively* compact neighbourhood U in A, i.e. a neighbourhood, such that its closure $\overline{U}$ is compact.

Definition 2. Let X be a topological space and F be a family of subsets of X. The family F is called *locally finite* if any point $x \in X$ has a neighbourhood which intersects finitely many elements of F.

Here are some topological results, which we will use later on.

Proposition 1.1. *Let X be locally compact Hausdorff space. Then a subset $A \subset X$ is locally compact if and only if A is locally closed.*

Proof. First, assume that A is a locally compact set and prove that it is locally closed. For any point $a \in A$ there is an open neighbourhood $U_a \subset A$ such that $\overline{U_a}^A$ is a compact set in X (we denote by $\overline{U}^A$ closure of U in A). Since U_a is open in X we can find a set $V \subset X$ open in X such that $U_a = A \cap V$.

We state $\overline{A} \cap V \subseteq A$. Suppose the statement is proven then $\overline{A} \cap V$ is the desired open neighbourhood of a in A and the openness of A in $\overline{A}$ is proven.

It remains to prove the statment. From the construction of $\overline{U_a}^A$ we see that $\overline{U_a}^A = \overline{U_a} \cap A = \overline{A \cap V} \cap A$ is compact in X. Since X is Hausdorff $\overline{U_a}^A$ is closed in X, therefore $\overline{\overline{A \cap V} \cap A} = \overline{A \cap V} \cap A$. The following is true

$$A \cap V \subseteq \overline{A \cap V} \cap A,$$

hence

$$\overline{A \cap V} \subseteq \overline{\overline{A \cap V} \cap A} = \overline{A \cap V} \cap A.$$

Since V is open $\overline{A} \cap V \subseteq \overline{A \cap V}$. Finally, we combine all together

$$\overline{A} \cap V \subseteq \overline{A \cap V} \cap A \subseteq A$$

and this completes the proof of the claim.

Second, assume A is locally closed. We want to find for any $a \in A$ a neighbourhood U_a such that $\overline{U_a}$ is compact.

We know for any point $a \in A \subset X$ there is a neighbourhood $V_a \subset X$ of a, such that $\overline{V_a}$ is compact in X. Set $U_a := \overline{V_a} \cap A$. Let W_α with $\alpha \in \Lambda$, where Λ is any set of indices, be an open cover of U_a, we want to find a finite subcover. Since A is open in $\overline{A}$, $W_\alpha \cap A$ is open in $\overline{A}$ for every $\alpha \in \Lambda$. Moreover $W_\alpha \cap A$ is open cover of $\overline{V_a} \cap A$. Now find a finite subcover W_i, where $i \in I$, and I is some finite subset of Λ, for $\overline{V_a}$, then $\overline{W_i} \cap A$ with $i \in I$ is a finite subcover for $\overline{V_a} \cap A$. □

Corollary 1.2. *Let X be a locally compact Hausdorff space. A dense subspace $D \subseteq X$ is locally compact if and only if it is open in X.*

Remark. If X is Hausdorff and locally compact then any point $x \in X$ has an open neighbourhood U and a neighbourhood V such that $\overline{V}$ is compact in U.

1.2 Stratified space

First we will formulate Whitney condition (b) for submanifolds of $\mathbb{R}^n$. Then we extend the definition to submanifolds of a manifold.

Definition 3. Let X, Y be smooth submanifolds of $\mathbb{R}^n$. A pair (X, Y) satisfies *Whitney condition b* for submanifolds of $\mathbb{R}^n$ at a point $y \in Y$ if the following holds. Let $x_n \in X$, $y_n \in Y$ with $n \in N$ be some sequences of points, such that $x_n \neq y_n$ for all $n \in N$ and $x_n \to x \leftarrow y_n$. Consider the sequence of tangent spaces T_{y_n} and suppose it converges to some r-plane $T_{y_n} \to \tau \subset \mathbb{R}^n$, where $r := \dim X$, assume also that the sequence of lines which go through origin and are parallel to the vectors $\frac{y_n - x_n}{|y_n - x_n|}$ converges in progective space $\mathbb{P}^{n-1}$ to a line $l \subset \mathbb{R}^n$; then $l \subset \tau$.

Here we state a lemma which will help us reformulate the definition for submanifolds of arbitrary manifold. Let (X', Y') be another pair of submanifolds of $\mathbb{R}^n$ and $y' \in Y'$.

Lemma 1.3. *Let U and U' be open neighbourhoods of y and y' correspondingly. Suppose there exist a diffeomorphism $\phi : U \to U'$ such that $\phi(U \cap X) = U' \cap X'$, $\phi(U \cap Y) = U' \cap Y'$ and $\phi(y) = y'$. Then (X, Y) satisfies condition b at y if and only if (X', Y') satisfies condition b at y'.*

Definition 4. Let M be a manifold and X, Y submanifolds and $y \in Y$. A pair (X, Y) satisfies *Whitney condition b* at y if for some coordinate chart (ϕ, U) with $y \in U$ the pair $(\phi(U \cap X), \phi(U \cap Y))$ satisfies condition b for submanifolds of $\mathbb{R}^n$ at $\phi(y)$.

Note that from Lemma 1.3 follows that if condition b holds in some chart than it holds in any.

Definition 5. A *Whitney stratified space (WSS)* W is a subset of a manifold M, for some μ, satisfying the following:

1) there is a locally finite *partition* $\mathscr{S}(W)$ (sometimes denoted as $\mathscr{S}$) of W into disjoint sets, i.e. $W = \bigsqcup_{X \in \mathscr{S}(W)} X$ where $\mathscr{S}(W)$ is a locally finite family of sets;

2) (*condition of frontier*) for any set $X \in \mathscr{S}(W)$ we have

$$\overline{X} \setminus X = \bigsqcup_{Y \in \mathscr{S}(W), \overline{X} \cap Y \neq \emptyset, Y \neq X} Y.$$

   We write $Y < X$ if Y is in the frontier of X

3) every set $X \in \mathscr{S}(W)$ is an embedded submanifold of $\mathbb{R}^\mu$ and called a *stratum*;

4) *(Whitney condition b)* every pair of strata satisfies Whitney condition b.

Definition 6. *Abstract stratified space (ASS)* is a triple $(W, \mathscr{S}, \mathcal{J})$ such that the following axioms are true

(A1) W is a locally compact Hausdorff space with countable basis of its topology;

(A2) $\mathscr{S}$ is a locally finite partition of W into locally closed sets;

(A3) every strata $X \in \mathscr{S}$ is a topological manifold with smoothness structure C^μ;

(A4) (*axiom of the frontier*) for every $X, Y \in \mathscr{S}$, such that $\overline{X} \cap Y \neq \emptyset$ we have $Y \subset \overline{X}$;

(A5) $\mathcal{J}$ is a set of triples $\{(T_X), (\pi_X), (\rho_X)\}$, where X is a stratum, T_X is an open neighbourhood of X in W, we call T_X *tubular neighbourhood*, $\pi_X : T_X \to X$ is a continuous retraction of T_X onto X (*local retraction*), $\rho_X : T_X \to [0; +\infty)$ is a continuous function (*tubular function*) and $\rho^{-1}(0) = X$;

(A6) let X, Y be any strata, define a set $T_{XY} := T_X \cap Y$ and maps

$$\pi_{XY} := \pi_X|_{T_{XY}} \text{ and } \rho_{XY} := \rho_X|_{T_{XY}},$$

then the mapping $(\pi_{XY}, \rho_{XY}) : T_{XY} \to X \times (0, +\infty)$ is a smooth submersion;

(A7) let X, Y be any strata then the following is true

$$\pi_{XY}\pi_{YZ}(v) = \pi_{XZ}(v) \quad (1.1)$$

$$\rho_{XY}\pi_{YZ}(v) = \rho_{XZ}(v), \quad (1.2)$$

for $v \in T_{XZ} \cap T_{YZ}$ and $\pi_{YZ} \in T_{XY}$.

Definition 7. Let $(W, \mathscr{S}, \mathcal{J})$ and $(W', \mathscr{S}', \mathcal{J}')$ be two ASS. They are called *equvalent*, if the following condition holds:

- $W = W'$ and $\mathscr{S} = \mathscr{S}'$ moreover the two smoothness structures on a stratum $X \in \mathscr{S} = \mathscr{S}'$ given by the different stratifications are the same;

- if $\mathcal{J} = \{(T_X), (\pi_X), (\rho_X)\}$, $\mathcal{J}' = \{(T'_X), (\pi'_X), (\rho'_X)\}$ then for every stratum there exists a neighbourhood $T''_X \subset T_X \cap T'_X$ of X such that

$$\rho_X|_{T''_X} = \rho'_X|_{T''_X} \text{ and } \pi_X|_{T''_X} = \pi'_X|_{T''_X}.$$

Let $(W, \mathscr{S}, \mathcal{J})$ be an abstract stratified space. We want to construct new abstract stratified space equivalent to the given one, such that the new tubular neighbourhoods satisfy some nice axioms. Before we do it we show that every ASS is metrizable.

Definition 8. A Hausdorff space is called *regular*, if each point and a closed set not containing the point have disjoint neighbourhoods.

Definition 9. A Hausdorff space is called *normal* if each pair of closed disjoint sets has disjoint neighbourhoods.

Theorem 1.4. *(J. Nagata and Yu.M. Smirnov) [1, Ch. IX] A topological space is metrizable if and only if it is regular and has a basis that can be decomposed into an at most countable collection of locally finite families.*

Since every locally compact space is regular (Dugundji [1]) every ASS is regular. Moreover every ASS has a countable basis for its topology, so we have the immediate corollaries.

Corollary 1.5. *Every ASS is metrizable.*

Corollary 1.6. *Every subset of an ASS is normal.*

Lemma 1.7. *Every ASS $(W, \mathscr{S}, \mathcal{J})$ is equivalent to ASS which satisfies the following axioms. Let X, Y be strata*

(B1) if $T_{XY} \neq \emptyset$, then $X < Y$;

(B2) if $T_X \cap T_Y \neq \emptyset$, then X and Y are comparable, i.e. one of the following holds $X < Y, Y < X$ or $X = Y$.

Proof. We will construct the new control data $\mathcal{J}'$ for $(W, \mathscr{S}, \mathcal{J})$ such that if two strata are not comparable, then $T'_X \cap T_Y = \emptyset$. Suppouse we found such system of tubular neighbourhoods. If $T'_X \cap T'_Y \neq \emptyset$, then X and Y are comparable(A11). The first axiom (A10) will follow from the construction of the tubular neighbourhoods.

We proceed by induction on number of strata.

In case there is only one stratum, there is nothing to prove. Assume we can construct the desired control data for the stratified space with k strata.

Let now $(W, \mathscr{S}, \mathcal{J})$ be ASS with $k+1$ strata. Let $X \in \mathscr{S}$ and consider $\mathscr{Y} := \cup_{Y \in \mathscr{S}, \overline{Y} \cap X = \emptyset, \overline{X} \cap Y = \emptyset} Y$. $\mathscr{Y}$ is a stratified space and the number of strata is less or equal k, by assumption there is a control data, which is satisfied the axioms. Denote it (T'_Y).

Set $W_X := W \setminus \partial X$. Note that W_X is open in W and $X \subset W_X, \mathscr{Y} \subset W_X$. Moreover

$$\overline{\mathscr{Y}} := \overline{\bigcup_{Y \in \mathscr{S}, \overline{Y} \cap X = \emptyset, \overline{X} \cap Y = \emptyset} Y} = \bigcup_{Y \in \mathscr{S}, \overline{Y} \cap X = \emptyset, \overline{X} \cap Y = \emptyset} \overline{Y}$$

is closed in W_X.

$$\overline{X} \cap W_X = (X \cup \delta X) \cap W_X = X$$

so X is also closed is W_X.

Any subset of ASS is normal by the Corollary 1.6. From normality of W_X follows that there are two open disjoint sets U_X and $U_{\mathscr{Y}}$ such that $X \subset U_X$ and $\mathscr{Y} \subset U_{\mathscr{Y}}$. Set $T''_X := T_X \cap U_X$ and $T''_Y = T'_Y$ for $Y \in \mathscr{Y}$. Denote $\pi'' = \pi|_{T''_X}$ and $\rho'' = \rho|_{T''_X}$ for all strata $X \in \mathscr{S}$.

□

1.3 Controlled vector fields

Definition 10. Let $(W, \mathscr{S}, \mathcal{J}))$ be a stratified space. A *stratified vector field* is a set $\{\eta_X$— smooth vector field on$X : X \in \mathscr{S}\}$.

Definition 11. A stratified vector field η on W is *controlled by* $\mathcal{J}$ if it satisfies the following *controlled conditions*: for any strata X, Y, where $X > Y$, there exists a neighbourhood T'_Y of Y in T_Y such that for each $v \in T'_Y \cap X$, we have

$$(\eta_X \rho_{YX})(v) = 0, \tag{1.3}$$

$$(\pi_{YX})_* \eta_X(v) = \eta_Y(\pi_{YX}(v)). \tag{1.4}$$

Definition 12. Let P be a smooth manifold and $f : V \to P$ a continuous mapping. f is a *controlled submersion* if the following conditions are satisfied for any stratum X

- $f|_X : X \to P$ is a smooth submersion;
- there is a neighbourhood T'_X of X in T_X such that $f(v) = f(\pi_X(v))$ for any $v \in T'_X$.

Proposition 1.8. *If $f : V \to P$ is a controlled submersion, then for any smooth vector field on P, there is a controlled vector field η on V such that $f_* \eta(v) = \zeta(f(v))$ for all $v \in V$.*

Proof. By induction on the dimension of V.

If $\dim V = 0$ the statement of the proposition is trivial.

Assume that the statement is true for any stratified space with dimension $\dim V \leq k$. Let now V be a stratifiad space and $\dim V = k+1$.

Consider *k-skeleton* of V

$$V_k := \bigsqcup_{Y \in \mathscr{S}, \dim Y \leq k} Y.$$

It can be easily seen that k-skeleton is a stratified space of dimension k, hence by inducton assumptoin there is a controlled vector field η_k on V_k such that

$$f_* \eta_k(v) = \zeta(f(v)) \quad \text{for any} v \in V_k.$$

We will show in two steps that there is a vector field η on V such that it is an extension of η_k, i.e. $\eta|_{V_k} = \eta_k|_{V_k}$, and $f_*\eta(v) = \zeta(f(v))$ for any $v \in V$.

1st Step. Fix a top stratum X, that is a startum with $\dim X = k+1$. Define

$$\mathscr{Y}_X := \bigsqcup_{Y \in \mathscr{S}, Y < X} Y,$$

clearly, for every $Y \in \mathscr{Y}_X$ we have $\dim Y < \dim X = k+1$, hence $\mathscr{Y}_X \subset V_k$. In the first step we will construct suitable control data for $\mathscr{Y}_X$.

By induction assumption η_k is controlled, so for any $Y \in \mathscr{Y}_X$ there is a neighbourhood T^1_Y of Y in T_Y such that for any starta Z, where $Z > Y$, the control conditions are satisfied

$$(\eta_Z \rho_{YZ})(v) = 0,$$

$$(\pi_{YZ})_* \eta_Z(v) = \eta_Y(\pi_{YZ}(v))$$

for any $v \in T^1_Y \cap Z$. Since f is controlled there is a neighbourhood T^2_Y of Y in T_Y such that

$$f(v) = f\pi_Y(v), \qquad \text{for all } v \in T^2_Y.$$

Set $T'_Y := T^1_Y \cap T^2_Y$ for all $Y \in V_k$.

Choose a neighbourhood T^3_Y of Y in T'_Y such that for any strata Z, where $Z \in V_k$ and $Y < Z$,

$$\pi_Z(T^3_Y \cap T^3_Z) \subseteq T'_Y. \tag{1.5}$$

By the axiom $(A11)$ there are tubular neighbourhoods (T^4_Y) for $Y \in V_k$ suth that if Y anf Z are not comparable, then $T^4_Y \cap T^4_Z = \emptyset$.

Set $T''_Y := T^3_Y \cap T^4_Y$ for all $Y \in V_k$.

We claim that there is a stratified vector field η_X on X satisfying the control conditions and the assertion of the proposition, namely

$$(\eta_X \rho_{YX})(v) = 0, \qquad \text{for all } v \in T''_Y \cap X, \tag{1.6}$$

$$(\pi_{YX})_* \eta_X(v) = \eta_Y(\pi_{YX}(v)), \qquad \text{for all } v \in T''_Y \cap X, \tag{1.7}$$

$$f_* \eta_X(v) = \zeta(f(v)), \qquad \text{for all } v \in X. \tag{1.8}$$

To prove this claim will clearly be enough to prove the proposition.

For a point $v \in X$ define a set $S_v := \{Y : Y < X, v \in T''_Y\}$. By the construction of (T''_Y), if Y and Z are not comparable, then $T''_Y \cap T''_Z = \emptyset$. Hence S_v is totally ordered set. If the S_v is not empty, then there is the maximal stratum Y_v, that is $Y_v > Y$ for any $Y \in S_v$.

Suppouse S_v is not empty and (1.6), (1.7) holds at v. Then (1.6), (1.7) holds for all $Y \in S_v$. If $Y < Y_v$ by the choice of T''_Y we have $\pi_{Y_v} \in T'_Y$ (see (1.5)). Then

$$\begin{aligned} \eta_X \rho_{YX}(v) = \eta_X \rho_{YY_v} \pi_{Y_v X}(v) = (\pi_{Y_v X})_* \eta_X(v) \rho_{YY_v} \\ \overset{(1.7)}{=} \eta_{Y_v}(\pi_{Y_v X}(v)) \rho_{YY_v} \overset{(1.6)}{=} 0 \end{aligned}$$

and

$$\begin{aligned} (\pi_{Y_v Y})_* \eta_X(v) \overset{(A9)}{=} (\pi_{YY_v})_* (\pi_{Y_v X})_* \eta(v) \overset{(1.7)}{=} (\pi_{YY_v})_* \eta_{Y_v}(\pi_{Y_v X}(v)) \\ \overset{(1.7)}{=} \eta_Y(\pi_{YY_v} \pi_{Y_v X}(v)) \overset{(A9)}{=} \eta_Y(\pi Y_v x(v)). \end{aligned}$$

Thus (1.6), (1.7) holds at v for all $Y \in S_v$. Furthermore

$$f_* \eta_X(v) = (f \circ \pi_{Y_v X})_* \eta_X(v) \overset{(1.7)}{=} f_* \eta_{Y_v}(\pi_{Y_v X}(v)) \overset{(1.8)}{=} \zeta(f(v)).$$

Thus 1.8 holds at v.

This shows that to construct η_X satisfying (1.6), (1.7), (1.8) for all $Y < X$, it is enough to construct η_X satisfying (1.6) and (1.7) for Y_v at v for all $v \in X$ for which S_v is nonempty, and satisfying (1.8) at v for all $v \in X$ for which S_v is empty. Clearly, we can construct a vector field η_X in a neighbourhood of each point v in X satisfying the appropriated conditions (1.6) and (1.7) or (1.8). Since the set of vectors satisfying the appropriated conditions in TX_v in convex, we may construct η_X globally by means of a partition of unity.

2nd Step. Before now we considered only one top stratum. The desired controlled vector field was constructed for this stratum. Since the condition that a vector field be controlled involves only strata, which are comparable and not equal, we may use the above method to construct the controlled vector fields for each top strata separately. $\square$

Proposition 1.9. *Let X be a stratum of an abstract stratified space W and ϵ be a smooth positive function on it, i.e. $\epsilon \in C^\infty(X, \mathbb{R}_>)$. Then $T_X^\epsilon = \{v \in T_X : \rho_X(v) < \epsilon(\pi_X(v))\}$ form a neighbourhood basis of X in T_X.*

Proof. We will proof the proposition by contradiction. Assume T_X^ϵ do not form a neighbourhood basis. Then there is a point $v \in X$, such that T_X^ϵ do not form a neighbourhood basis of the point.

Since W is locally compact we can find a neighbourhood U_v of v with compact closure. Consider $T_v^\epsilon := U_v \cap T_X^\epsilon$. From the assumption follows that there is an open neighbourhood V of v such that a set $T_v^{\epsilon_n} \setminus V$ is not empty for infinitely many functions ϵ_n, where $n \in \mathbb{N}$. Take a sequence of points $y_n \in T_v^{\epsilon_n} \setminus V$. Every point of it lies in compact set U_v, consequently there is a converging subsequence $\{y_{n_k}\}$ with limit $y \in T_X$, where $n_k \in \mathbb{N}$. We know from construction of T_X^ϵ that $\pi_X(y) = \lim_{n\to\infty} \pi_X(y_{n_k}) = v$ and $\rho_X(y) = \lim_{n\to\infty} \rho_X(y_{n_k})$=0. Hence $y = v$. But we assumed that in the neighbourhood V of x there is no points form $\{y_n\}$. The contradiction proves the proposition. $\square$

Proposition 1.10. *Let $X_\epsilon := \{(x,t) : x \in X, 0 \le t \le \epsilon\}$. In the notations of the previous proposition there is a smooth function ϵ, such that a map $\sigma_X^\epsilon := (\pi_X^\epsilon, \rho_X^\epsilon) : T_X^\epsilon \to X^\epsilon$ is proper.*

Proof. Let K_ϵ be a compact set in X_ϵ it can be written in the following form $K_\epsilon := K \times k$, where $K \subset X$ is compact and $k \subset [0, \epsilon]$ is compact. Since V is locally compact we can choose locally finite covering of K by sets V_α with compact closure. Moreover we can choose finite subcover by V_i, where $0 < i < k$ and $i, k \in \mathbb{N}$. The set $\pi_X^{-1}(K) \cap \rho_X^{-1}(k)$ is closed because the map (π_X, ρ_X) is continuous. Now take ϵ such that $\pi_X^{-1}(K) \cap \rho_X^{-1}(k) \subset \cup V_i$ with $0 < i < k$. The union of finitely many compact sets V_i is compact. So we have that $\pi_X^{-1}(K) \cap \rho_X^{-1}(k)$ is closed and is a subset of compact set, hense it is compact.

$\square$

1.4 Local one-parameter group

Let V be a locally compact space.

Definition 13. A *local one-parameter group* on V is a pair (J, α), where J is an open subset of $\mathbb{R} \times V$ and $\alpha : J \to V$ is a continuous mapping such that the following conditions are satisfied

(a) $0 \times V \subseteq J$ and $\alpha(0, v) = v$ for all $v \in V$;

(b) if $v \in V$, then the set $J_v := J \cap (\mathbb{R} \times v) \subseteq \mathbb{R}$ is an open interval (a_v, b_v), possibly infinite at one or both ends;

(c) if $v \in V$ and $s, t+s \in (a_v, b_v)$ and $t \in (a_{\alpha(s,v)}, b_{\alpha(s,v)})$, then $\alpha(t+s, v) = \alpha(t, \alpha(s, v))$;

(d) for any $v \in V$ and any compact set $K \subseteq V$, there exists $\epsilon > 0$ such that $\alpha(t, v)$ does not lie inside K if $t \in (a_v, a_v + \epsilon) \cup (b_v - \epsilon, b_v)$.

Proposition 1.11. *Every controlled vector field on V generates a unique local one parameter group (J, α).*

Proof. From the definition of stratified vector field we know that the restriction η_X of η to X is a smooth vector field on X. Applying a result from differential geometry we see that η_X generates a unique smooth local one-parameter group (J_X, α_X) of diffeomorphisms of X. Set

$$J = \bigcup_{X \in \mathscr{S}} J_X,$$

$$\alpha = \bigcup_{X \in \mathscr{S}} \alpha_X.$$

A pair (J, α) is a local one-parameter group generated by η. We see that (a), (b) and (c) in the definition of local one-parameter group hold, and if (J, α) is a local one-parameter group then it is generated by η. Since each pair (J_X, α_X) is unique, (J, α) is unique. We need to proof condition (d) and that J is open, α is continuous.

Proof the condition (d).

We proof by contradiction. Assume that (d) does not hold. Then there exists $v \in V$ and a compact set $K \subset V$, such that $\alpha(t, v) \in K$ for values of t arbitrary close to a_v or b_v. Consider the case where $\alpha(t, v) \in K$ for values arbitrary close to b_v, the other case is analogous.

Choose a sequence $\{t_i\}_{i \in \mathbb{N}}$ that converges to b_v from below such that $x = \lim \alpha(t_i, v) \in K$. Let X, Y denote two stratum such that $x \in X$ and $v \in Y$. Since x is a limit point of points from Y we conclude that $X \subset \overline{Y}$. Suppose $X = Y$.

$$x = \lim \alpha(t_i, v) = \alpha(b_v, v) = \alpha_Y(b_v, v) \in Y,$$

hence $b_v \in J_Y \cap (\mathbb{R} \times v)$, but this contradicts with the definition of number b_v, i.e. with the fact $J_Y \cap (\mathbb{R} \times v)) = (a_v, b_v)$. We conclude that $X < Y$.

Since $x \in X$ is a limit point of sequence $\alpha(t_i, v)$ we can choose a number $N \in \mathbb{N}$ such that all points $\alpha(t_n, v)$ for $n > N$ lie an a tubular neighbourhood of X, i.e. $\alpha(t_n, v) \in T_X$ and $\rho_{XY}(\alpha(t_n, v)), \pi_{XY}(\alpha(t_n, v))$ are defined. Also the control conditions hold

$$(\eta_Y \rho_{XY})(\alpha(t_n, v)) = 0;$$
$$(\pi_{XY})_* \eta_Y(\alpha(t_n, v)) = \eta_X(\pi_{XY}(\alpha(t_n, v))).$$

Set $x_n := \pi_{XY}(\alpha(t_n, v))$ and note that $x_n \in X$. By taking n large enough choose $\epsilon > b_v - t_n$ such that $[0, \epsilon] \subseteq J_{x_n} = J_X \cap (\mathbb{R} \times x_n)$. We can do so because by the definition of local one-parameter group of diffeomorphisms of X, $0 \times X \subseteq J_X$ and $J_X \cap (\mathbb{R} \times z)$ is open for any $z \in X$.

Since ρ_{XY} is a continuous function, $\rho_{XY}(x) = 0$ and $\lim\limits_{n \to \infty} \alpha(t_n, v) = x$ we see $\lim\limits_{n \to \infty} \rho_{XY}(\alpha(t_n, v)) = 0$. Hence for some positive number ϵ_X the inequality $\rho_{XY}(\alpha(t_n, v)) < \epsilon_X$ holds on $\alpha([0, \epsilon], x_n)$, and control conditions are satisfied for the following set

$$M_{[0,\epsilon]} := \{m \in Y : \rho_{XY}(m) = \rho_{XY}(\alpha(t_n, v)) \text{ and } \pi_{XY}(m) = \alpha([0, \epsilon], x_n)\}.$$

Set M is compact, because $\rho_{XY}(\alpha(t_n, v)) < \epsilon_X$ on $\alpha([0, \epsilon], x_n)$, the sets $\alpha(t_n, v)$ and $\alpha([0, \epsilon], x_n)$ are compact and the map (π_{XY}, ρ_{XY}) is proper by Proposition 1.10 on $T_X^{\epsilon_X}$. From control conditions follows that the curves, along which function ρ_{XY} is constant, are tangent to vector filed η. Also we kmow that $\frac{d}{dt}\alpha_Y(t, v) = \eta_Y(\alpha_Y(t, v))$ so we conclude that for any $s \in [0, \epsilon]$ points $\alpha(t_n + s, v)$ lie in the following set

$$M_s := \{m \in Y : \rho_{XY}(m) = \rho_{XY}(\alpha(t_n, v)) \text{ and } \pi_{XY}(m) = \alpha(s, x_n)\}.$$

Since $b_v > \epsilon + t_i$ we extended the domain (a_v, b_v). It is a contradiction to the assumption that $\alpha(t_i, v)$ converges to x as i tends to infinity. We proved condition (d).

Proof of openness of J and continuity of α.

Choose arbitrary point $(t, v) \in J$ and suppose $t \in [0, \infty)$, in other cases proof is analogous. We find an open neighbourhood of (t, v) such that it lie in J and then prove that α is continuous.

Let X be the stratum where v lies. Since X is locally compact there is a neighbourhood U of v such that $\overline{U}$ is compact. By the definition of local parameter group we find a small positive number ϵ such that $[-\epsilon, t + \epsilon] \times \overline{U} \subseteq J$. For any stratum local parameter group on it is continuous, hence $\alpha_X([-\epsilon, t + \epsilon] \times \overline{U})$ is compact.

Define a set

$$\Sigma := \{y \in T_X : \rho_X(y) \leq \epsilon_1 \text{ and } \pi_X(y) \in \alpha_X([-\epsilon, t + \epsilon] \times \overline{U})\}$$

. By the Proposition 1.10 we can choose ϵ_1 such that the map (π_X, ρ_X) is proper on $T_X^{\epsilon_1}$, hence Σ is compact.

Let Y be a stratum which contains $y \in \Sigma$. Controll conditions hold for $y \in \Sigma$ (we may choose smaller ϵ_1 if it is needed). Note that the following set

$$\Sigma_0 := \{y \in T_y : \rho_X(y) < \epsilon_1 \text{ and } \pi_X(y) \in U\} \subset \Sigma$$

is a neighbourhood of v in V. From control conditions follows that the function ρ_X remains constant along the path $\alpha(t, y)$ for any $y \in \Sigma$, i.e.

$$\rho_X(\alpha(s, y)) = \rho_X(y),$$

and the projection π_X commute with $\alpha(s, y)$ in the following sense

$$\pi_X(\alpha(s, y)) = \alpha(s, \pi_X(y)).$$

for all $s \in J_y$ such that $\alpha(s_1, y) \in \Sigma$ for $0 \leq s_1 < s$. From these facts and (d), it follows that $(-\epsilon, t + \epsilon) \times \Sigma_0 \subseteq J$. Thus J contains a neighbourhood of (t, v).

We have shown that if $(t', y) \in (t - \epsilon, t + \epsilon) \times \Sigma_0$, then $y' := \alpha(t', y) \in T_X$, $\rho(y') \leq \epsilon_1$ and $\pi_X(y') = \alpha(t', \pi_X(y))$. Hence, for an arbitrary small neighbourhood of $\alpha(t, v)$ we may choose $\epsilon > 0$ and a neighbourhood Σ_1. Consecuently, α is continuous at (t, v). □

Corollary 1.12. *(J.Mather [2]) Let P be a manifold and $f : V \to P$ be a proper, controlled submersion. Then f is locally trivial fibration.*

Definition 14. A topological space is called *locally path-connected* if for any point x and an open neighbourhood U_x there is an open neighbourhood $V_x \subset U_x$, which is path-connected in a subspace topology.

Proposition 1.13. *Every compact stratified space is locally path connected.*

Proof. Let $x \in W$ be any point. We want to find a path connected neighbourhood. Let X be a stratum such that $x \in X$ and U_x be a compact path connected neighbourhood of x in W. We can finde such neighbourhood because X is a topological manifold, hence locally path connected. Define

$$A := \overline{T_X} \cap \pi_X^{-1}(U_x)$$

Since W is compact and $\overline{T_X}$ is closed, $\overline{T_X}$ is compact. Moreover $\pi_X^{-1}(U_x)$ is compact. Set A is compact as an intersection of two compact sets.

Assume now that A is not path connected, i.e. there is a point $y \in A$ sich that it cannot be connected with x by a path. Denote by $A_y \subset A$ path connected component for y.

By construction $A_y \cap U_x = \emptyset$. Define

$$d := \inf\{\rho_X(y) : y \in A_y\}.$$

Since A is compact and A_y is closed in A, A_y is compact. Hence there is a sequence of points $y_n \in A_y, n \in \mathbb{N}$ with $\rho(y_n) \to d$ as $n \to \infty$ and a point $\tilde{y} \in A_y$ such that $\rho(\tilde{y}) = d$.

Let Y be a stratum $y \in Y$ and $Y > X$. By the axion $(A6)$ from the defenition of a abstract stratified set $(\pi_{XY}, \rho_{XY}) : T_X \cap Y \to X \times (0; +\infty)$ is a smooth submersion than there is a point $y' \in Y \cap A_y$ with $\rho(y') < d$ and $\pi_X(y') = \pi_X(\tilde{y})$. this contradicts with assumotion that d is infimum.

Now we will construct a path explicitly. Let y be a point from the neighbourhood $Y \cap \dot{T}_X^{\epsilon_X}$ of x and let construct a path between the points x and y. We now that

$$\rho_X : Y \cap \dot{T}_X^{\epsilon_X} \to (0, \epsilon_X) \quad \text{is a submersion.}$$

Moreover $\pi^{-1}(y) = x \in U_x$. From the Corollary 1.12 follows that $\pi_X^{-1} \cap \dot{T}_X^{\epsilon_x} \approxeq (0, \epsilon_x) \times U_x \times L_X$, where $L_X := \pi_X^{-1}(x) \cap \rho_X^{-1}(\epsilon)$ is a fiber over 0.

$$\rho_X(y) = \rho \in (0, \epsilon_x)$$

Set

$$c(t) := (t\rho, x, l), \quad \text{where} t \in (0, 1).$$

and put $c(0) := y$.

□

2 Simplicial complexes

2.1 Defenitions and examples

Definition 15. A *simplicial complex* K consists of a set v of *verticies* a set s of finite nonempty subsets of v called *simplicies* such that

- any vertex is a simplex;
- any nonempty subset of a simplex is a simplex.

Definition 16. A simplex s containing exactly $q + 1$ verticies is called *q-simplex*, we write $dim(s) = q$.

Definition 17. If $s' \subset s$, s' is called a *face* of s (if $s' \neq s$ *proper face*).

Remark. The simplicies of K are partially ordered by the face relation ($s' \leq s$ if s' is a face of s).

Example 2.1. Let A be any set then the set of all finite nonempty subsets of A is a simplicial complex.

Example 2.2. Let s be a simplex of a simplicial complex K, the set of all proper faces of s is a simplicial complex denoted by $\dot{s}$.

Example 2.3. Let K be a simplicial complex. A set $K^q = \{s \subset K | \dim s \leq q\}$ is called a q-skeleton of K

2.2 Realization of a simplicial complex in $\mathbb{R}^n$

Given a nonempty simplicial complex K, let $|K|$ be set of all functions $\alpha : \{v\} \to I$ such that

a) for each α set $\{v \in K : \lambda(v) \neq 0\}$ is a simplex of K;

b) for each α $\sum_{v \in K} \alpha(v) = 1$.

If $K = \emptyset$ we define $|K| = \emptyset$.

Definition 18. The real number $\alpha(v)$ is called th vth *barycentric coordinate* of α.

$$d(\alpha, \beta) := \sqrt{\sum_{v \in K} (\alpha(v) - \beta(v))^2}$$

defines a metric on $|K|$ and the topology on $|K|$ defined by this metric is called the *metric topology.* We denote be $|K|_d$ set $|K|$ with the metric toplogy.

Definition 19. For a simplex $s \in K$ the *closed simplex* $|s|$ is defined by

$$|s| := \{\alpha \in |K| : \alpha(v) \neq 0 => v \in s\}.$$

Note that if s is a q-simplex, then $|s|$ is in one-to-one correspondence with the following set

$$\{x \in \mathbb{R}^{q+1} : 0 \leq x_i \leq 1, \sum_{i=0}^{q} x_i = 1\}.$$

The metric topology on $|K|_d$ induces on $|s|$ a topology and the topological space $|s|_d$ is homeomorphic to $\{x \in \mathbb{R}^{q+1} : 0 \leq x_i \leq 1, \sum_{i=0}^{q} x_i = 1\}$. If s_1, s_2

are two simplicies then $s_1 \cap s_2$ is either empty set, then also $|s_1| \cap |s_2| = \emptyset$, or a face of both s_1 and s_2, then $|s_1 \cap s_2| = |s_1| \cap |s_2|$. Hence the set $|s_1| \cap |s_2|$ is closed in both sets $|s_1|_d$ and $|s_2|_d$. The topologies induced on $|s_1| \cap |s_2|$ from $|s_1|_d$ and from $|s_2|_d$ are equal. So there is a toplogy on $|K|$ coherent with $\{|s|_d : s \in K\}$, we will call it *weak topology.* $|K|$ will denote the space with weak topology.

Definition 20. For a simplex $s \in K$ the *open simplex* is defined by

$$< s >:= \{\alpha \in |K| : \alpha(v) \neq 0 <=> v \in s\}.$$

Since $< s >= |s| \setminus |\dot{s}|$, $< s >$ is an open subset of $|s|$, but it should not be open in $|K|$.

Proposition 2.4. *A finite simplicial complex is a WSS.*

Proof. Let K be a finite simplicial complex with n number of its verticies and let $|K|$ be its realization in $\mathbb{R}^\mu$. Open simplicies constitute a partition of $|K|$. Indeed, every point $\alpha \in K$ belongs to a unique open simplex, namely the open simplex $< s >$, where $s = \{v \in K : \alpha(v) \neq 0\}$. We will show that $|K|$ satisfy properties $1) - 4)$ of WSS.

1) Number of open simplicies of $|K|$ is equal to 2^n, so we have finite partition of simplicial complex into disjoint open simplicies.

2) We know that $|s| \setminus < s > = |\dot{s}|$, where s is any simplex of K, $|s|$ is a closed simplex in $|K|$, $< s >$ is open simplex in $|K|$, $\dot{s}$ is a set of al proper faces of s. This equality gives us precisely the condition of frontirer for $|K|$.

3) By the definition of realization of K every simplex s is an embedded submanifold of $\mathbb{R}^\mu$.

4) To show Whitney condition b take two open simplecies s_1, s_2 and siquences of points $x_n \in s_1, y_n \in s_2$ such that $x_n \to x \leftarrow y_n$. Tangent space T_{y_n} is the same for all points y_n, so $T_{y_n} = \tau$ for all $n \in \mathbb{N}$. Vectors $\frac{x_n - y_n}{|x_n - y_n|}$ lie in space τ, which is Euclidian space, so the limit of the vectors also lies in τ. □

References

[1] J. Dugundji, *Topology*, Allyn and Bacon, Inc., Boston, 1966.

[2] J. Mather *Notes on topological stability*, volume 49, number 4 of Bulletin (New series) of the American Mathematical Society, pages 475-506, 2012.